

One-loop $f(R)$ Gravitational Modified Models

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Abstract. The one-loop quantisation of a general class of modified gravity models around a classical de Sitter background is presented. Application to the stability of the models is addressed.

1. Introduction

In contrast with what expected by the so called standard cosmological model, quite surprisingly recent astrophysical data indicate that the Universe is currently in a phase of accelerated expansion. This has been seen by the observation of the light curves of several hundred of type I^a supernovae [1] and also emerges by a detailed analysis of the Cosmic Microwave Background as recently measured by WMAP satellite and other experiments [2].

The associated theoretical issue, called the Dark Energy problem, might be solved in different ways. The simplest one consists in the introduction of suitable new cosmological “matter” fields with unusual equation of state (quintessence, phantom, k-essence). Another possibility consists in the modification of General Relativity by adding new gravitational higher order derivative and non linear terms in the curvature invariants to the Einstein-Hilbert action. Finally, the accelerated expansion of the Universe can also be accounted for within the framework of non-perturbative renormalisation of quantum gravity [3].

In this paper, we will be mainly interested in this second possibility. A short motivation may be presented as follows: the inclusion of terms which grow up when curvature decreases, for example inverse curvature terms [4], which may origin from string/M-theory [5], may explain such current accelerated expansion and may give negligible contributions to early cosmology. However, such modified gravity models with Einstein-Hilbert term plus inverse curvature terms contain some instabilities [6] and do not pass solar system tests, but their further modification by higher derivative curvature squared terms could improve their behaviour [7]. In this respect, it has to be note that in general such kind of models are not renormalisable and for this reason they have to be seen not as fundamental, but as effective theories. In such cases the models can be made renormalisable, but the prise to pay is the lack of unitarity [8, 9]. Furthermore, if Einstein’s gravity is only an effective theory, then at the early Universe the (effective) quantum gravity should be different from Einstein’s one.

The widely discussed possibility in this direction is quantum R^2 gravity (for a review, see [10]). However, other modifications deserve attention, because they may

produce extra terms, which may help to realise the early time inflation. This is supported by the possibility of accelerated expansion with simple modified gravity.

Here, we will report the general $f(R)$ gravity at one-loop level in de Sitter Universe [11]. Such a program for the case of Einstein's gravity has been initiated in [12, 13, 14] (see also [15]). Also in that case the theory is multiplicatively non-renormalisable.

Using generalised zeta-function regularisation (see, for example [16, 17]), it is possible to get the one-loop effective action and to study the possibility of stabilisation of de Sitter background by quantum effects. Moreover, such an approach may suggest the way how to resolve the cosmological constant problem [14]. Hence, the study of one-loop $f(R)$ gravity is a quite natural step in the realisation of such a program, having in mind, that consistent treatment of quantum gravity does not exist yet.

2. Models for Dark Energy

As mentioned in the Introduction, it is possible to accommodate the current cosmic acceleration without modify the Einstein-Hilbert action, but in such a case one has to introduce by hands a cosmic fluid with negative pressure, realizing in this way a naive model for the Dark Energy component.

It is also well known that the simplest theoretical possibility of this kind consists in the introduction, in the standard cosmological model, of a *positive cosmological constant*. It is a historical fact that the cosmological constant was introduced for the first time in GR by Einstein himself, in order to get a static cosmological model and immediately abandoned after the observation of cosmic expansion. Now it appears again but, in some sense, for the opposite reason.

As it is well known, within the framework of quantum field theory in curved space-time, a constant cosmological term can be identified with the vacuum energy, but unfortunately, the estimation of such a quantity gives rise to a value extremely large if compared with the one expected for the cosmological constant. In fact, within the quantum field theory, the vacuum energy is divergent. A cutoff at the Planck or electroweak scale leads to a cosmological constant which is, respectively, 10^{123} or 10^{55} times larger than the observed value, $\Lambda/8\pi G \simeq 10^{-47} \text{ GeV}^4$. This is the well known *cosmological constant problem*, which probably might be solved in a unified theory of all interactions.

Other possibility consists in using a cosmological scalar field, a sort of “dynamical cosmological constant”, as for example *quintessence*, if the rate pressure/density is greater than -1 or *phantom* if such a rate is less than -1 . We remind that for the cosmological constant the rate pressure/density is exactly -1 .

3. The $f(R) = R - \mu^4/R$ model

Now let us discuss the second possibility, i.e. the fact that cosmic acceleration can also be explained by adding other non linear terms to the Einstein-Hilbert Lagrangian. For example, it is well known that quadratic terms in the curvature, may be induced by quantum effects associated with conformally coupled matter fields and these effects modify the cosmological solutions of Einstein's equations at *early time*. In this way one could obtain, for example, a de Sitter inflationary phase [18].

On the contrary, by considering terms depending on the inverse of the curvature, one modifies the solutions at *present time* [4]. In this case, one has to pay attention and try to modify GR in such a way that astrophysical tests are not violated, since

GR is in excellent agreement with astrophysical data and one has to save also early cosmology, which is in good agreement with observations.

We illustrate this second possibility by considering the simplest model introduced in [4] and defined by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{\mu^4}{R} \right) + S_{matter}.$$

This is the action in the so called *matter frame* and this gives rise to 4th order field equations.

By making use of a suitable conformal transformation, one may pass to the so called *Einstein frame*, in which the action assumes the Einstein-Hilbert form and the gravitational additional degrees of freedom are represented by a scalar field ϕ with a complicated potential, which reads

$$V(\phi) = \frac{\mu^2}{8\pi G} \left(e^{-\phi/\sqrt{12\pi G}} - 1 \right)^{1/2} e^{-\phi/\sqrt{3\pi G}}.$$

Depending on the initial values of ϕ one can have different solutions. For a special value of $\phi(0)$, one has a de Sitter solution, but this is unstable and requires fine tuning in order have corrections to standard cosmology starting at the present epoch. Moreover, it is in contrast with gravitational tests on solar system. For this reason, different models have been considered.

4. Arbitrary $f(R)$ modified gravitational models

The starting point is the classical action depending on a generic function of the scalar curvature R

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R),$$

which gives rise to the field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Delta) f'(R) = 0.$$

Requiring the existence of solutions with constant scalar curvature $R = \hat{R}$, one obtains the conditions [19]

$$2f(\hat{R}) = \hat{R} f'(\hat{R}), \quad R_{\mu\nu} = \frac{f(\hat{R})}{2f'(\hat{R})} g_{\mu\nu} = \frac{\hat{R}}{4} g_{\mu\nu}.$$

and this means that these solutions are Einstein's spaces with an effective cosmological constant

$$\Lambda_{eff} = \frac{f(\hat{R})}{2f'(\hat{R})} = \frac{\hat{R}}{4}.$$

Such a class of constant curvature solutions contains black holes in the presence of a non vanishing cosmological constant[20], like the Schwarzschild-(anti)de Sitter and all the topological solutions associated with a negative Λ_{eff} [21]. In general, their black hole entropies do not satisfy the area law [20].

5. Quantum field fluctuations around a maximally symmetric space

In this Section, we will present a summary of the one-loop quantisation of the general modified models in the Euclidean signature. We shall make use of the background field method and zeta-function regularisation, having in mind that, in general, these models are not renormalisable and one is dealing only with an effective approach.

To begin with, we recall that the condition

$$2\hat{f} = \hat{R}\hat{f}', \quad \hat{f} = f(\hat{R}), \quad \hat{f}' = f'(\hat{R}),$$

ensures the existence of constant curvature solutions and in particular maximally symmetric spaces solutions like for, example, the de Sitter space, the one we are interested in.

For maximally symmetric spaces, the Riemann and Ricci tensors are given by the expressions

$$\hat{R}_{ijrs} = \frac{\hat{R}}{12} (\hat{g}_{ir}\hat{g}_{js} - \hat{g}_{is}\hat{g}_{jr}), \quad \hat{R}_{ij} = \frac{\hat{R}}{4} \hat{g}_{ij}, \quad R = \hat{R}.$$

Now we expand the metric and all quantities in the action around the maximally symmetric solution, that is

$$g_{ij} = \hat{g}_{ij} + h_{ij}, \quad g^{ij} = \hat{g}^{ij} - h^{ij} + h^{ik}h_k^j + \dots \quad h = \hat{g}^{ij}h_{ij},$$

and up to second order in h_{ij}

$$\begin{aligned} \frac{\sqrt{g}}{\sqrt{\hat{g}}} &= 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{ij}h^{ij} + \mathcal{O}(h^3) \\ R &\sim \hat{R} - \frac{\hat{R}}{4}h + \nabla_i \nabla_j h^{ij} - \Delta h + \frac{\hat{R}}{4}h^{jk}h_{jk} \\ &\quad - \frac{1}{4}\nabla_i h \nabla^i h - \frac{1}{4}\nabla_k h_{ij} \nabla^k h^{ij} + \nabla_i h_k^i \nabla_j h^{jk} - \frac{1}{2}\nabla_j h_{ik} \nabla^i h^{jk}. \end{aligned}$$

Here the symmetric tensor h_{ij} has to be considered as a small fluctuation about the background metric \hat{g}_{ij} .

For technical reasons, it is convenient to carry out the standard expansion of the tensor field h_{ij} in irreducible components, namely

$$h_{ij} = \hat{h}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \nabla_i \nabla_j \sigma + \frac{1}{4}g_{ij}(h - \Delta \sigma),$$

where σ is the scalar component, while ξ_i and \hat{h}_{ij} are the vector and tensor components with the properties

$$\nabla_i \xi^i = 0, \quad \nabla_i \hat{h}^i_j = 0, \quad \hat{h}^i_i = 0.$$

In terms of the irreducible components of the h_{ij} field, the quadratic part of the Lagrangian density, disregarding total derivatives, becomes

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{12} \hat{h}^{ij} (3\hat{f}\Delta - 3\hat{f}' + \hat{R}\hat{f}') \hat{h}_{ij} \\ &\quad + \frac{1}{16} (2\hat{f} - \hat{R}\hat{f}') \xi^i (4\Delta + \hat{R}) \xi_i \\ &\quad + \frac{1}{32} h \left[9\hat{f}'' \Delta^2 - 3(\hat{f}' - 2\hat{R}\hat{f}'') \Delta + 2\hat{f} - 2\hat{R}\hat{f}' + \hat{R}^2 \hat{f}'' \right] h \\ &\quad + \frac{1}{32} \sigma \left[9\hat{f}'' \Delta^4 - 3(\hat{f}' - 2\hat{R}\hat{f}'') \Delta^3 \right] \end{aligned}$$

$$\begin{aligned}
& -(6\hat{f} - 2\hat{R}\hat{f}' - \hat{R}^2\hat{f}'') \Delta^2 - \hat{R}(2\hat{f} - \hat{R}\hat{f}') \Delta \Big] \sigma \\
& + \frac{1}{16} h \left[-9\hat{f}'' \Delta^3 + 3(\hat{f}' - 2\hat{R}\hat{f}'') \Delta^2 + \hat{R}(\hat{f}' - \hat{R}\hat{f}'') \Delta \right] \sigma.
\end{aligned}$$

In order to quantise the model, now one has to add *gauge fixing* and *ghost contributions*. Such terms are quite complicated, but with the help of a tensor manipulations program, we were able to obtain the one-loop effective action (here written in the Landau gauge) (on-shell condition: $X = (2\hat{f} - \hat{R}\hat{f}')/4 = 0$)

$$\begin{aligned}
\Gamma_{off-shell} &= \frac{24\pi\hat{f}}{G\hat{R}^2} + \frac{1}{2} \log \det \left(-\Delta_2 - \frac{\hat{R}}{6} \frac{X + 2\hat{f}}{X - 2\hat{f}} \right) \\
& - \frac{1}{2} \log \det \left(-\Delta_1 - \frac{\hat{R}}{4} \right) - \frac{1}{2} \log \det \left(-\Delta_0 - \frac{\hat{R}}{2} \right) \\
& + \frac{1}{2} \log \det \left\{ \left(-\Delta_0 - \frac{5\hat{R}}{12} - \frac{X - 2\hat{f}}{6\hat{R}\hat{f}''} \right)^2 \right. \\
& \quad \left. - \left[\left(\frac{5\hat{R}}{12} + \frac{X - 2\hat{f}}{6\hat{R}\hat{f}''} \right)^2 - \frac{\hat{R}^2}{6} - \frac{X - \hat{f}}{3\hat{f}''} \right] \right\}, \\
\Gamma_{on-shell} &= \frac{24\pi\hat{f}}{G\hat{R}^2} + \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_2 + \frac{R_0}{6} \right) \right] \\
& - \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_1 - \frac{\hat{R}}{4} \right) \right] \\
& + \frac{1}{2} \log \det \left[\ell^2 \left(-\Delta_0 - \frac{\hat{R}}{3} + \frac{2\hat{f}}{3\hat{R}\hat{f}''} \right) \right]. \tag{5.1}
\end{aligned}$$

On the de Sitter manifold (more precisely $SO(4)$, since we work in the Euclidean section) the eigenvalues of the Laplace operator are known and this means that zeta-functions are exactly computable and so we can obtain the one-loop effective action in the closed form

$$\Gamma = \Gamma(\hat{R}) = \frac{24\pi}{G\hat{R}^2} f(\hat{R}) + F_1(\hat{R}) + F_2(\hat{R}) \log \frac{\ell^2 \hat{R}}{12}.$$

Here $F_1(\hat{R})$, $F_2(\hat{R})$ are complicated, but known functions of the scalar curvature \hat{R} .

6. Stability of de Sitter solution in $f(R)$ models

The one-loop effective action may be used to investigate the role of quantum corrections of these modified gravitational models to the background cosmology (see, for example, [11]). In the following, we would like to present an application to the stability of such models.

In fact, the stability of the de Sitter solution may be obtained by imposing the one-loop effective action to be real and this happens if the differential Laplace-like operators, which determine the effective action, do not possess negative eigenvalues. The eigenvalues of Laplace-like operators in $SO(4)$ of the kind $L_i = -\Delta_i + c_i \hat{R}$, can be evaluated, recalling that for the pure Laplacians one has

- $-\Delta_0$: $(n^2 + 3n)(\hat{R}/12)$, $n = 0, 1, 2, \dots$
- $-\Delta_1$: $(n^2 + 5n + 3)(\hat{R}/12)$, $n = 0, 1, 2, \dots$
- $-\Delta_2$: $(n^2 + 7n + 8)(\hat{R}/12)$, $n = 0, 1, 2, \dots$

Then, from equation (5.1), we get the following conditions, which state the stability of de Sitter solution:

$$2f(\hat{R}) - \hat{R}f'(\hat{R}) = 0, \quad \frac{f(\hat{R})}{2f'(\hat{R})} > 0,$$

$$\frac{2f(\hat{R})}{\hat{R}^2 f''(\hat{R})} = \frac{f'(\hat{R})}{\hat{R} f''(\hat{R})} > 1.$$

The first two equations ensure the existence of a solution with positive constant curvature, while the third one ensures the stability of such a solution. Such a condition has been obtained in [23] by a classical perturbation method.

For example, for the model

$$f(R) = R - \frac{\mu^4}{R},$$

one has

$$f'(R) = 1 + \frac{\mu^4}{R^2}, \quad f''(R) = -2\frac{\mu^4}{R^3},$$

and $\hat{R} = \sqrt{3}\mu^2$. As a result, the model is always unstable.

For the slightly modified model [22]

$$f(R) = R - \frac{\mu^4}{R} + aR^2,$$

one has again $\hat{R} = \sqrt{3}\mu^2$, but now

$$f'(R) = 1 + \frac{\mu^4}{R^2} + 2aR, \quad f''(R) = -2\frac{\mu^4}{R^3} + 2a.$$

A direct calculation leads to the stability condition $a > 1/3\sqrt{3}\mu^2$, in agreement with [5, 23].

Finally in the case

$$f(R) = R + aR^2 - 2\Lambda,$$

one has $\hat{R} = 4\Lambda$, and

$$f'(R) = 1 + 2aR, \quad f''(R) = 2a.$$

As a result, the model is stable for $a > 0$ [19].

7. Conclusions

Generalising a previous program concerning the one-loop Einstein's gravity in the de Sitter background [14], we have here presented the one-loop effective action for a general $f(R)$ gravitational modified model. This one-loop effective action may be used to investigate the role of quantum corrections in cosmology.

Furthermore, as a non trivial application, we have derived the condition which ensures the stability of the de Sitter solution in such a class of modified gravity theories. Such a condition is in full agreement with the one obtained in [23], where the covariant and gauge-invariant formalism of Bardeen-Ellis-Bruni-Hwang [24] has been used.

We have also seen that generalising the simplest model in [4], it is possible to build up models with a stable de Sitter solution, which in principle could explain the recent cosmological data. From this point of view, models depending on a suitable function of the Gauss-Bonnet invariant seem more promising, since they pass solar system tests for any reasonable choice of the function [25].

References

- [1] A.G. Riess *et al.*, *Astron. Astrophys.* **116**, 1009 (1998); *Astron. J.* **118**, 2668 (1999); *Astrophys. J.* **560**, 49 (2001); *Astrophys. J.* **607**, 665 (2004); S. Perlmutter *et al.*, *Nature* **391**, 51 (1998); *Astrophys. J.* **517**, 565 (1999); J.L. Tonry *et al.*, *Astrophys. J.* **594**, 1 (2003); R. Knop *et al.*, *Astrophys. J.* **598**, 102 (2003); B. Barris *et al.*, *Astrophys. J.* **602**, 571 (2004).
- [2] A.D. Miller *et al.*, *Astrophys. J. Lett.* **524**, L1 (1999); P. de Bernardis *et al.*, *Nature* **400**, 955 (2000); A.E. Lange *et al.*, *Phys. Rev. D* **63**, 042001 (2001); A. Melchiorri, L. Mersini, C.J. Odman and M. Trodden, *Astrophys. J. Lett.* **536**, L63 (2000); S. Hanany *et al.*, *Astrophys. J. Lett.* **545**, L5 (2000); D.N. Spergel *et al.*, *Astrophys. J. (Suppl.)* **148**, 175 (2003); C.L. Bennett *et al.*, *Astrophys. J. (Suppl.)* **148**, 1 (2003); T.J. Pearson *et al.*, *Astrophys. J.* **591**, 556 (2003); A. Benoit *et al.*, *Astron. Astrophys.* **399**, L25 (2003).
- [3] A. Bonanno and M. Reuter, *Int. J. Mod. Phys. D* **13** (2004) 107. A. Bonanno, G. Esposito, G. Rubano and P. Scudellaro, “The accelerated expansion of the universe as a crossover phenomenon,” arXiv:astro-ph/0507670.
- [4] S. Capozziello, S. Carloni and A. Troisi, Recent Research Developments in Astronomy and Astrophysics-RSP/AA/21-2003, astro-ph/0303041; S. M. Carroll, V. Duvvuri, M. Trodden and M. Turner, *Phys. Rev. D* **70**, astro-ph/0306438.
- [5] S. Nojiri, S. D. Odintsov, *Phys. Letters B* **576** (2003) 5, hep-th/0307071;
- [6] T. Chiba, *Phys.Lett. B* **575** (2003) 1, astro-ph/0307338; A.D. Dolgov and M. Kawasaki, *Phys.Lett. B* **573** (2003) 1, astro-ph/0307285; M.E. Soussa and R.P. Woodard, *Gen.Rel.Grav.* **36** (2004) 855, astro-ph/0308114.
- [7] S. Nojiri and S.D. Odintsov, *Phys.Rev. D* **68** (2003) 123512, hep-th/0307288; hep-th/0412030; E. Abdalla, S. Nojiri and S.D. Odintsov, hep-th/0409177.
- [8] K.S. Stelle, *Phys. Rev. D* **16** (1977) 953.
- [9] N. H. Barth and S. M. Christensen, *Phys. Rev. D* **28** (1983) 1876.
- [10] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro. *Effective Action in Quantum Gravity* IOP Publishing, Bristol, 1992.
- [11] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, *JCAP* **0502** (2005) 010 [arXiv:hep-th/0501096].
- [12] G. W.Gibbson and M.J. Perry, *Nucl. Phys. B* **146** (1978) 90.
- [13] S.M. Christensen and M.J. Duff, *Nucl. Phys. B* **170** (1980) 480.
- [14] E.S. Fradkin and A.A. Tseytlin, *Nucl. Phys. B* **234** (1984) 472.
- [15] S.D. Odintsov, *Europhys.Lett.* **10** (1989) 287; *Theor.Math.Phys.* **82** (1990) 66; T.R. Taylor and G. Veneziano, *Nucl. Phys. B* **345** (1990) 210;
- [16] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini. *Zeta regularization techniques with applications* World Scientific, 1994.
- [17] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, *Phys.Reports.* **269** (1996)1.
- [18] A. Starobinsky, *Phys.Lett. B* **91** (1980) 99; S.G. Mamaev and V.M. Mostepanenko, *JETP* **51** (1980) 9; B. Geyer, S. D. Odintsov and S. Zerbini, *Phys. Letters B* **460** (1999) 58.
- [19] A.A. Starobinsky, *Sov. Phys. - JETP Lett.* **34**, 438 (1981); J.D. Barrow and A.C. Ottewill, *J. Phys. A: Math. Gen.* **16** (1983) 2757.
- [20] I. Brevik, S. Nojiri, S. D. Odintsov and L. Vanzo, *Phys. Rev. D* **70**, 043520 (2004) [arXiv:hep-th/0401073].
- [21] L. Vanzo, *Phys. Rev. D* **56**, 6475 (1997) [arXiv:gr-qc/9705004].
- [22] R. Dick, *Gen. Rel. Grav.* **36**, 217 (2004), gr-qc/0307052.
- [23] V. Faraoni, *Phys. Rev. D* **72**, 061501 (2005) [arXiv:gr-qc/9705004].
- [24] J.M. Bardeen, *Phys. Rev. D* **22**, 1882 (1980); G.F.R. Ellis and M. Bruni, *Phys. Rev. D* **40**, 1804 (1989); G.F.R. Ellis, J.-C. Hwang and M. Bruni, *Phys. Rev. D* **40**, 1819 (1989); G.F.R. Ellis, M. Bruni and J.-C. Hwang, *Phys. Rev. D* **42**, 1035 (1990).
- [25] S. Nojiri and S. D. Odintsov, arXiv:hep-th/0508049; to appear in *Phys. Lett. B*.